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THE BURR XII EXPONENTIATED WEIBULL MODEL

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Abstract. A new extension of the exponentiated Weibull model with variable shapes of hazard rates is introduced and studied. Some of its mathematical properties are derived. Two applications are allocated to show the importance and flexibility of the new model.

1. Introduction and motivation

A random variable (RV) Z is said to have the Exponentiated Weibull [EW (a, b)] distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$\mathbf{h}_{a,b}(z) = abz^{b-1}\left(-e^{-z^{b}}+1\right)^{a-1} e^{-z^{b}},$$

and

$$\mathbf{H}_{a,b}(z) = \left(-e^{-z^{b}}+1\right)^{a},$$

respectively, for $z > 0$, $\alpha > 0$ and $\beta > 0$, when $\alpha = 1$ we get the the one parameter W model. In this work, we will introduce and study a new version of the EW model called the Burr XII EW model based on the BXII-G family intridused by (see [5]) and give an overall description of its properties along with two real data applications. The new EW model is motivated by its important flexibility in applications. By means of two applications, it noticed that the BXIIEW model provides better fits than other W models. Following [5], the CDF of the new can be written as

$$F(x) = 1 - \underbrace{\left\{1 + \left[\frac{\left(1-e^{-x^{b}}\right)^{a}}{1-\left(1-e^{-x^{b}}\right)^{a}}\right]^{\alpha}\right\}^{-\beta}}_{A}, \tag{1}$$

and its corresponding PDF is given by

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$$f(x)=\alpha\beta abx^{b-1}\,e^{-x^{b}}\,\frac{\left(1-e^{-x^{b}}\right)^{a\alpha-1}}{\left[1-\left(1-e^{-x^{b}}\right)^{a}\right]^{\alpha+1}}\left\{1+\left[\frac{\left(1-e^{-x^{b}}\right)^{a}}{1-\left(1-e^{-x^{b}}\right)^{a}}\right]^{\alpha}\right\}^{-(1+\beta)}, \tag{2}$$

the corresponding hazard rate function (HRF) can be expressed as

$$h(x)=\alpha\beta abx^{b-1}\,e^{-x^{b}}\,\frac{\left(1-e^{-x^{b}}\right)^{a\alpha-1}}{\left[1-\left(1-e^{-x^{b}}\right)^{a}\right]^{\alpha+1}}\left\{\left[\left(1-e^{-x^{b}}\right)^{-a}-1\right]^{-\alpha}+1\right\}^{-1}.$$

We can provide an easy transformation of an uniform RV to generate the BXIIEW RV, in fact, if U is a uniform RV in $(0,1)$ and

$$W=\left[-1+(1-U)^{-\beta^{-1}}\right]^{\alpha^{-1}},$$

then

$$X=\left\{-\ln\left[1-\left(\frac{W}{1+W}\right)^{a^{-1}}\right]\right\}^{b^{-1}},$$

has CDF (1). A physical interpretation of the BXIIEW model can be given as follows. Let Y be a lifetime RV having EW distribution. The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x will be

$$\xi(X)=\left(1-e^{-x^{b}}\right)^{a}/\left[1-\left(1-e^{-x^{b}}\right)^{a}\right].$$

The function $\zeta(X)$ is always monotonic and non-decreasing. If we are interested in modeling the randomness of the odds by the RV T having the BXIIEW as in (5), we can write

$$\Pr\left\{\left[\left(1-e^{-x^{b}}\right)^{-a}-1\right]^{-1}\geq T\right\}=1-\left\{1+\left[\left(1-e^{-x^{b}}\right)^{-a}-1\right]^{-\alpha}\right\}^{-\beta}=F(x).$$

which is identical to (1). So, if X has the BXIIEW model, then $T=\xi(X)$ has the BXIIEW CDFgiven by (1).

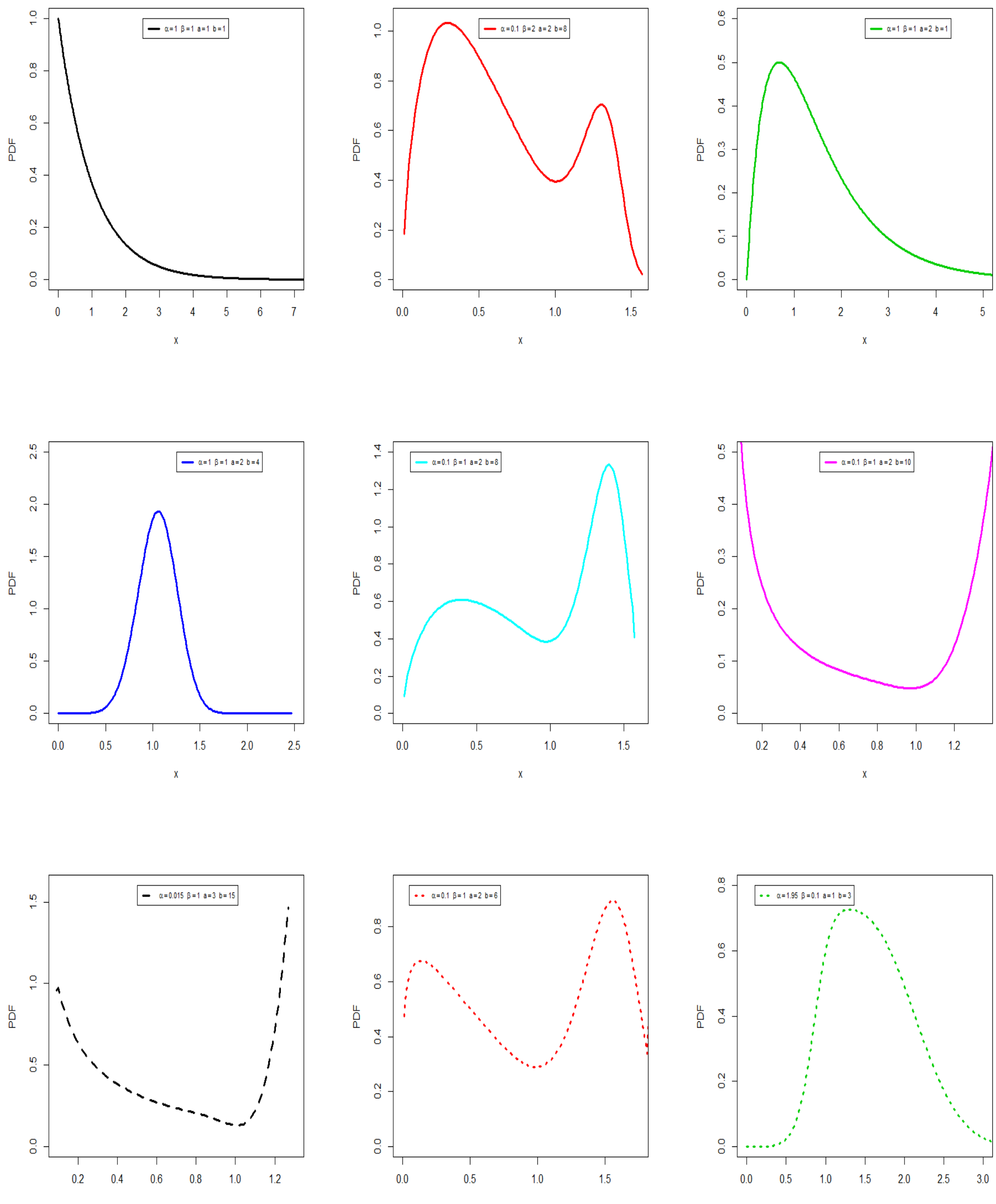

Figure 1: Plots of the BXIIEW PDF .

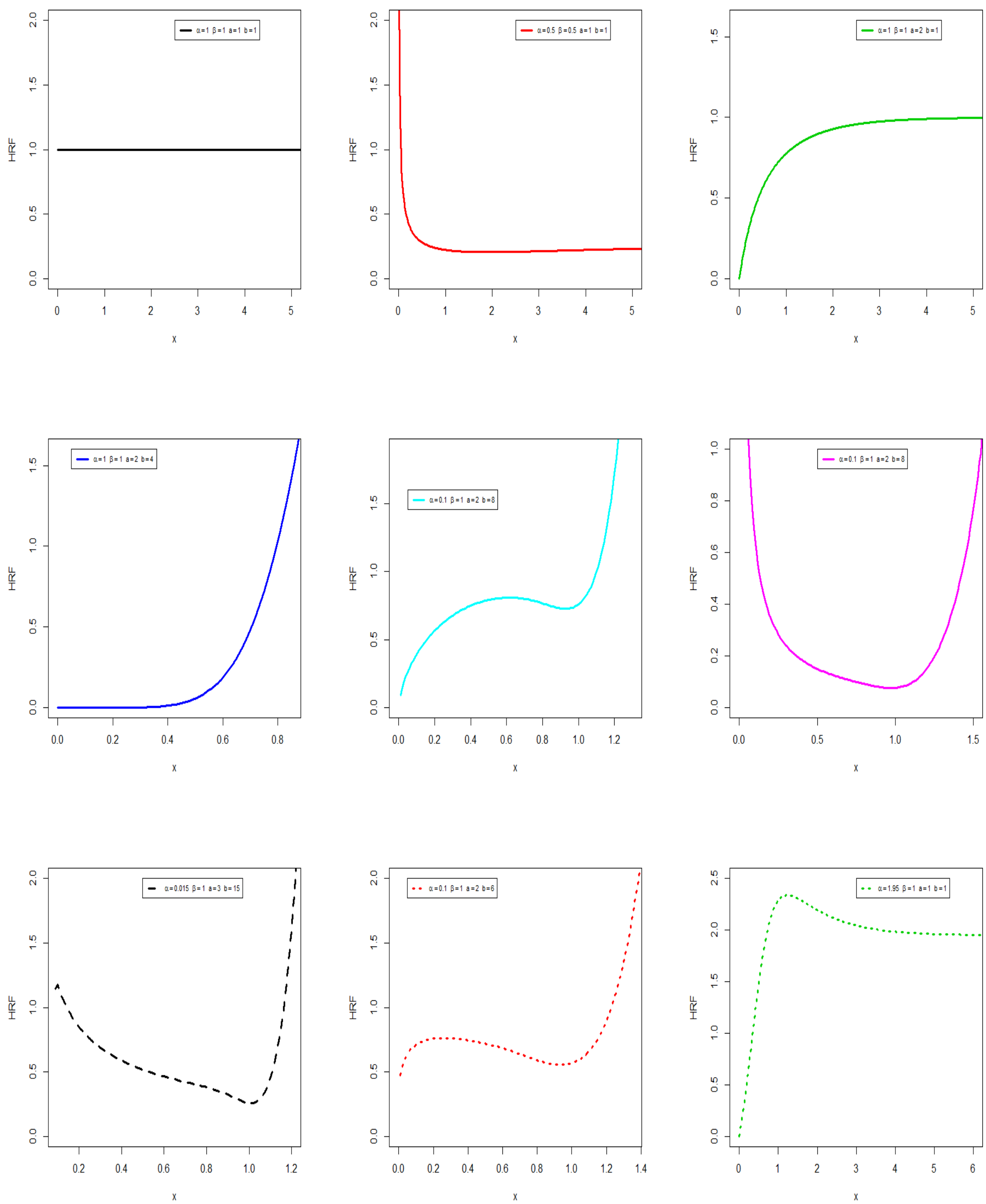

Figure 2: Plots of the BXIIEW HRF.

From Figure 1 we conclude that the PDF of the BXIIEW distribution exhibits all important shapes like symmetric, left skewed, right skewed and bimodal, from Figure 2 we conclude that the HRF of the BXIIEW distribution exhibits bathtub, constant, increasing, decreasing, unimodal then constant and unimodal then bathtub hazard rates.

This paper is organized as follows. In Section 2, we derive some of mathematical properties for the BXIIEW model. Maximum likelihood estimation for the BXIIEW parameters is addressed in Section 3. In Section 4, potentiality of the proposed model is illustrated by means of two real data sets. Finally, Section 5 ends with some conclusions.

2. Properties

2.1. **Linear representation.** First, we consider two power series

$$(1+\tau)^{-a}=\sum_{m=0}^{\infty}2^{-a-m}\,(-1+\tau)^{m}\binom{-a}{m} \tag{3}$$

and

$$(1-\tau)^{-a}\,|_{(|\tau|<1)}^{(a>0)}=\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{k!\,\Gamma(a)}\,\tau^{k}. \tag{4}$$

Applying (3) for A in (1) gives

$$F(x)=1-\sum_{k=0}^{\infty}2^{-\beta-k}\binom{-\beta}{k}\left\{\left[\frac{\left(1-e^{-x^{b}}\right)^{a}}{1-\left(1-e^{-x^{b}}\right)^{a}}\right]^{\alpha}-1\right\}^{k}.$$

Second, using the binomial expansion, the last equation can be expressed as

$$\begin{aligned}F(x) &= 1-\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(-1)^{i}}{2^{\beta+k}}\left(1-e^{-x^{b}}\right)^{(k-i)a\alpha}\\ &\quad\times\underbrace{\left[1-\left(1-e^{-x^{b}}\right)^{a}\right]^{-(k-i)\alpha}}_{B}\binom{k}{i}\binom{-\beta}{k}.\end{aligned}$$

Third, applying (4) for B in the last equation gives

$$F(x)=1-\sum_{j,k=0}^{\infty}\sum_{i=0}^{k}q_{i,j,k}\mathbf{H}_{(k-i)\alpha+j,b}(x), \tag{5}$$

where

$$\mathbf{H}_{a\gamma,b}(x)=[\mathbf{H}_{a,b}(x)]^{\gamma}=\mathbf{H}_{a\gamma,b}(x)=\left(1-e^{-x^{b}}\right)^{a\gamma}$$

is the cdf of the EW model with power parameter $a\gamma$ where $a\gamma>0$ and

$$q_{i,j,k}=\frac{(-1)^{i}\,\Gamma\left([k-i]\alpha+j\right)}{2^{\beta+k}\,j!\,\Gamma\left([k-i]\alpha\right)}\binom{k}{i}\binom{-\beta}{k}.$$

Upon differentiating (5), we obtain

$$f(x)=\sum_{\substack{j,k=0\\ j+k\geq 1}}^{\infty}\sum_{i=0}^{k}c_{i,j,k}\ \mathbf{h}_{(k-i)\alpha+j,b}\left(x\right), \tag{6}$$

where $\mathbf{h}_{a\gamma,b}(x)$ denotes the EW density with power parameter $a\gamma>0$ and $c_{i,j,k}=-q_{i,j,k}$.

2.2. **Moments and generating function.** The r^{th} ordinary moment of X say $\mu_r'=\mathbf{E}(X^r)$, is determined from (6) as

$$\mu_r'\ |_{(r>-b)}^{(j+k\geq 1)}=\Gamma\left(1+rb^{-1}\right)\sum_{j,k,h=0}^{\infty}\sum_{i=0}^{k}c_{i,j,k,h},$$

where

$$c_{i,j,k,h}=c_{i,j,k}\ c_h^{((k-i)\alpha+j,r)},$$

$$c_h^{(\tau,r)}=\frac{\tau\left(-1\right)^h}{\left(h+1\right)^{(r+b)/b}}\binom{\tau-1}{h}$$

and

$$\Gamma\left(1+\zeta\right)|_{(\zeta\in\mathbb{R}^+)}=\zeta!=\zeta\times\left(\zeta-1\right)\times\left(\zeta-2\right)\times...\times 1=\prod_{w=0}^{\zeta-1}\left(\zeta-w\right),$$

and

$$\int_0^{\infty}x^{\zeta-1}\,e^{-t}dx=\Gamma\left(\zeta\right)$$

is the complete gamma function. The r^{th} incomplete moment of X, say $\mathbf{I}_r(t)$, can be determined from (6) as

$$\mathbf{I}_r(t)|_{(r>-b)}^{(j+k\geq 1)}=\int_{-\infty}^{t}x^r\,f(x)dx=\gamma\left(1+rb^{-1},t^{-b}\right)\sum_{j,k,h=0}^{\infty}\sum_{i=0}^{k}c_{i,j,k,h}\ . \tag{7}$$

where $\gamma\left(\zeta,q\right)$ is the incomplete gamma function.

$$\begin{aligned}\gamma\left(\zeta,q\right)|_{(\zeta\neq 0,-1,-2,...)} &= \int_0^q t^{\zeta-1}\exp\left(-t\right)dt\\ &= \frac{q^{\zeta}}{\zeta}\left\{{}_1\mathbf{F}_1\left[\zeta;\zeta a+1;-q\right]\right\}\\ &= \sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{k!\left(\zeta+k\right)}q^{\zeta+k},\end{aligned}$$

and ${}_1\mathbf{F}_1\left[\cdot,\cdot,\cdot\right]$ is a confluent hypergeometric function. The moment generating function (MGF) $M(t)=\mathbf{E}(\mathrm{e}^{\mathrm{t\,X}})$ of X follows from (6) as

$$M(t)|_{(r>-b)}^{(j+k\geq 1)}=\Gamma\left(1+rb^{-1}\right)\sum_{j,k,h,r=0}^{\infty}\sum_{i=0}^{k}\left(t^r c_{i,j,k,h}/r!\right).$$

2.3. **Probability weighted moments (PWMs).** The $(s,r)^{th}$ PWM of X denoted by $\rho_{s,r}$ is formally defined by

$$\rho_{s,r} = \mathbf{E}\{X^s\, F(X)^r\} = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\, dx.$$

Using (1), we have

$$F(x)^r = \left(1 - \left\{1 + \left[\frac{\left(1-e^{-x^b}\right)^a}{1-\left(1-e^{-x^b}\right)^a}\right]^\alpha\right\}^{-\beta}\right)^r.$$

Expanding z^λ in Taylor series, we can write

$$z^\tau = \sum_{h=0}^{\infty} \frac{(\tau)_h}{h!}\,(z-1)^h = \sum_{i=0}^{\infty} z^i \zeta_i(\tau), \tag{8}$$

where

$$(\tau)_h = \tau(\tau-1)(\tau-2)\ldots(\tau-h+1)$$

is the descending factorial and

$$\zeta_i(\tau) = \sum_{h=i}^{\infty} \frac{(-1)^{h-i}}{h!}\,(\tau)_h \binom{h}{i}.$$

First, applying the Taylor series in z^λ for $F(x)^r$, we obtain

$$F(x)^r = \sum_{i=0}^{\infty} (-1)^i\, \zeta_i\,(r) \left\{1 + \left[\frac{\left(1-e^{-x^b}\right)^a}{1-\left(1-e^{-x^b}\right)^a}\right]^\alpha\right\}^{-i\beta}.$$

Second, using (2) and the last equation, we have

$$\begin{aligned} f\,(x)\;F(x)^r \;&=\; \alpha\beta ab x^{b-1}\,\zeta_i\,(r)\,e^{-x^b}\left(1-e^{-x^b}\right)^{a-1} \frac{\left[\left(1-e^{-x^b}\right)^a\right]^{\alpha-1}}{\left[1-\left(1-e^{-x^b}\right)^a\right]^{\alpha+1}} \\ &\quad \times \sum_{i=0}^{\infty} (-1)^i \underbrace{\left\{1 + \left[\frac{\left(1-e^{-x^b}\right)^a}{1-\left(1-e^{-x^b}\right)^a}\right]^\alpha\right\}^{-(i+1)\beta-1}}_{C}. \end{aligned}$$

Applying (3) for C in the last equation, we obtain

$$f(x)\ F(x)^r = \alpha\beta ab \sum_{i,k=0}^{\infty} (-1)^i \zeta_i(r)\, x^{b-1} e^{-x^b} \left(1-e^{-x^b}\right)^{a-1} \times \frac{\left[\left(1-e^{-x^b}\right)^a\right]^{\alpha-1}}{\left[1-\left(1-e^{-x^b}\right)^a\right]^{\alpha+1}} 2^{-(i+1)\beta-k-1} \times \underbrace{\left\{-1+\left[\frac{\left(1-e^{-x^b}\right)^a}{1-\left(1-e^{-x^b}\right)^a}\right]^{\alpha}\right\}^k}_{D} \binom{-(i+1)\beta-1}{k}.$$

Third, using the binomial expansion for D, the last equation be rewritten as

$$f(x)\ F(x)^r = \alpha\beta abx^{b-1} e^{-x^b}\left(1-e^{-x^b}\right)^{a-1} \times \sum_{i,k=0}^{\infty}\sum_{j=0}^{k} (-1)^{i+j}\binom{k}{j}\binom{-(i+1)\beta-1}{k} \times \frac{\zeta_i(r)}{2^{(i+1)\beta+k+1}}\left[\left(1-e^{-x^b}\right)^a\right]^{(k-j+1)\alpha-1} \times \underbrace{\left[1-\left(1-e^{-x^b}\right)^a\right]^{-[(k-j+1)\alpha+1]}}_{E}.$$

Applying (4) for E in the last equation gives

$$f(x)\ F(x)^r = \sum_{i,k,m=0}^{\infty}\sum_{j=0}^{k} \alpha\beta \frac{(-1)^{i+j}\,\Gamma([k-j+1]\alpha+m+1)}{2^{(i+1)\beta+k+1}\, m!\,\Gamma([k-j+1]\alpha+1)} \times \frac{\zeta_i(r)}{\{a[\alpha(k-j+1)+m]\}}\binom{k}{j}\binom{-(i+1)\beta-1}{k} \times a[\alpha(k-j+1)+m]\, bx^{b-1} e^{-x^b}\left(1-e^{-x^b}\right)^{a[\alpha(k-j+1)+m]-1}$$

and then

$$f(x)\ F(x)^r = \sum_{k,m=0}^{\infty}\sum_{j=0}^{k} b_{j,k,m}^{(r)}\ \mathbf{h}_{a[\alpha(k-j+1)+m],b}(x), \qquad (9)$$

where $b_{j,k,m}^{(r)} = \alpha\,\beta\,\zeta_i(r)\,u_{j,k,m}$, $\zeta_i(r)$ is defined in (8) and (for $j \le k$)

$$u_{j,k,m} = \sum_{i=0}^{\infty} \frac{(-1)^{i+j}\ ([k-j+1]\alpha+1)^{(m)}}{2^{(i+1)\beta+k+1} m!\,\{a[\alpha(k-j+1)+m]\}}\binom{k}{j}\binom{-(i+1)\beta-1}{k},$$

where

$$c^{(w)} = \Gamma(c+w)/\Gamma(c)$$

denotes the rising factorial. Finally, the $(s,r)^{th}$ PWM of X can be determined from the EW moments as

$$\rho_{s,r}|_{(s>-b)}^{(j\leq k)} = \Gamma\left(1+sb^{-1}\right)\sum_{k,m,h=0}^{\infty}\sum_{j=0}^{k}\tau_{j,k,m,h}^{(a[\alpha(k-j+1)+m],s)}.$$

where

$$\tau_{j,k,m,h}^{(a[\alpha(k-j+1)+m],s)} = b_{j,k,m}^{(r)}\ c_{h}^{(a[\alpha(k-j+1)+m],s)}$$

2.4. **Residual life and reversed residual life functions.** The n^{th} moment of the residual life, say

$$\tau_n(t)\mid_{(X>t)}^{(n=1,2,\ldots)} = \mathbf{E}\left(X-t\right)^n.$$

The n^{th} moment of the residual life of X is given by

$$\tau_n(t)\mid_{(X>t)}^{(n=1,2,\ldots)} = \left[1-F(t)\right]^{-1}\int_t^{\infty}(x-t)^n dF(x).$$

Therefore

$$\tau_n(t)\mid_{(X>t),(n>-b)}^{(n=1,2,\ldots),(j+k\geq 1)} = \left[1-F(t)\right]^{-1}\sum_{j,k,h=0}^{\infty}\sum_{i=0}^{k}c_{i,j,k}^{(\tau)}\ c_h^{((k-i)\alpha+j,n)}\Gamma\left(1+nb^{-1},t^{-b}\right),$$

where

$$c_{i,j,k}^{(\tau)} = c_{i,j,k}\sum_{r=0}^{n}\left(1-t\right)^n,$$

$$\Gamma\left(\zeta,q\right)|_{(x>0)} = \int_q^{\infty}t^{\zeta-1}e^{-t}dt,$$

and

$$\Gamma\left(\zeta,q\right)+\gamma\left(\zeta,q\right) = \Gamma\left(\zeta\right)$$

The n^{th} moment of the reversed residual life, say

$$\omega_n(t)\mid_{(X\leq t,t>0)}^{(n=1,2,\ldots)} = \mathbf{E}(t-X)^n,$$

We obtain

$$\omega_n(t)\mid_{(X\leq t,t>0)}^{(n=1,2,\ldots)} = \left[F(t)\right]^{-1}\int_0^{t}(t-x)^n dF(x).$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$\omega_n(t)\mid_{(X\leq t,t>0),(n>-b)}^{(n=1,2,\ldots),(j+k\geq 1)} = \left[F(t)\right]^{-1}\sum_{j,k=0}^{\infty}\sum_{i=0}^{k}c_{i,j,k}^{(\omega)}c_h^{((k-i)\alpha+j,n)}\gamma\left(1+nb^{-1},t^{-b}\right),$$

where

$$c_{i,j,k}^{(\omega)} = c_{i,j,k}\sum_{r=0}^{n}\left(-1\right)^r t^{n-r}\binom{n}{r}.$$

2.5. **Moments of order statistics.** Let $X_1, \cdots, X_n$ be a random sample (RS) from the BXIIEW distributionl and let $X_{1:n}, \cdots, X_{n:n}$ be the corresponding order statistics. The PDF of the i^{th} order statistic, say $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} F^{r+i-1}(x),$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. From (9) we can write

$$f(x)\, F(x)^{r+i-1} = \sum_{k,m=0}^{\infty} \sum_{j=0}^{k} b_{j,k,m}^{(r+i-1)} \ \mathbf{h}_{a[\alpha(k-j+1)+m],b}(x),$$

where $b_{j,k,m}^{(r+i-1)}$ is defined there. So, the PDF of $X_{i:n}$ follows using the last expression as

$$f_{i:n}(x) = \frac{1}{\mathrm{B}(i, n-i+1)} \sum_{k,m=0}^{\infty} \sum_{r=0}^{n-i} \sum_{j=0}^{k} (-1)^r \binom{n-i}{r} b_{j,k,m}^{(r+i-1)} \ \mathbf{h}_{a[\alpha(k-j+1)+m],b}(x). \tag{10}$$

Based on (10), the moments of $X_{i:n}$ can be expressed as. Then we have

$$\mathbf{E}(X_{i:n}^q)\,|_{(q>-b)}^{(j\leq k)} = \frac{\Gamma\left(1+\frac{q}{b}\right)}{\mathrm{B}(i, n-i+1)} \sum_{k,m,h=0}^{\infty} \sum_{r=0}^{n-i} \sum_{j=0}^{k} \tau_{j,k,m,r,h}^{(a[\alpha(k-j+1)+m],q)},$$

where

$$\tau_{j,k,m,r,h}^{(a[\alpha(k-j+1)+m],q)} = (-1)^r \binom{n-i}{r} b_{j,k,m}^{(r+i-1)} \ c_h^{(a[\alpha(k-j+1)+m],q)}$$

2.6. **Quantile spread (QS) ordering.** The QS of a probability model describes how the probability mass is placed symmetrically about its median (Medi(X)) and hence can be used to formalize concepts like tailweight traditionally associated with kurtosis (Kur(X)) and peakedness (Peak(X)). The QS of a RV $X \sim$BXIIEW(α, β, a, b) with cdf in (1) is given by

$$QS_X(\vartheta) = \left[F^{-1}(\vartheta)\right] - \left[F^{-1}(1-\vartheta)\right] \ \forall\ \vartheta \in (0.5, 1)$$

and this implies

$$QS_X(\vartheta) = \left[S^{-1}(1-\vartheta)\right] - \left[S^{-1}(\vartheta)\right],$$

where $S^{-1}(1-\vartheta) = F^{-1}(\vartheta)$ and $1 - F = S$ is the survival function. It allows us to separate concepts of Kur(X) and Peak(X) for asymmetric models. Let X_1 and X_2 be two random variables follow LiW model with quantile spreads QS_{X_1} and QS_{X_2}, respectively. Then X_1 is called smaller than X_2 in quantile spread order, denoted as $X_1 \leq_{(QS)} X_2$, if $QS_{X_1}(\vartheta) \leq QS_{X_2}(\vartheta)$, $\forall\ \vartheta \in (0.5, 1)$. The following properties of the quantile spread order can be determined

- The order $\leq_{(QS)}$ is location-free, i.e.,

$$X_1 \leq_{(QS)} X_2 \text{ if } (X_1 + C) \leq_{(QS)} X_2 \text{ for any real } C.$$

- The order $\leq_{(QS)}$ is dilative which means

$$X_1 \leq_{(QS)} \zeta X_1 \text{ whenever } \zeta \geq 1 \text{ and } X_2 \leq_{(QS)} \zeta X_2\ ,\ \forall\ \zeta \geq 1.$$

- Assume F_{X_1} and F_{X_2} are symmetric, then

$$X_1 \leq_{(QS)} X_2 \text{ if, and only if } F_{X_1}^{-1}(\vartheta) \leq F_{X_2}^{-1}(\vartheta),\ \forall\ \vartheta \in (0.5, 1).$$

• The order $\leq_{(QS)}$implies ordering of the mean absolute deviation around the median, $\mathrm{MAD}(X_i)|_{(i=1,2)}$,

$$\mathbf{E}\left[|-\mathrm{Medi}(X_1)+X_1|\right]=\mathbf{MAD}(X_1)$$

and

$$\mathbf{E}\left[|-\mathrm{Medi}(X_2)+X_2|\right]=\mathbf{MAD}(X_2),$$

i.e.,

$$X_1 \leq_{(QS)} X_2 \ \Rightarrow \ MAD(X_1) \leq_{(QS)} \mathbf{MAD}(X_2).$$

• Finally

$$X_1 \leq_{(QS)} X_2 \text{ if, and only if } \ -X_1 \leq_{(QS)} -X_2.$$

3. Maximum Likelihood Estimation (MLE)

The MLEs enjoy some desirable statistical properties which can be used for confidence intervals (CI) and test statistics. The normal approximations for those estimators in large sample theory is easily to be handled analytically and numerically as well. We determine the MLEs of the parameters of the BXIIEW model only from complete samples. Other further works may be addressed using different methods for estimating the BXIIEW parameters such as least squares, moments, weighted least squares, Anderson-Darling, Jackknife, bootstrap, Cramér-von-Mises, Bayesian analysis, among others, and compare the estimators based on these methods. Let $x_1, \cdots, x_n$ be a RS from the BXIIEW distribution with parameters α, β, a and b. Let $\boldsymbol{\varphi}=(\alpha, \beta, a, b)^T$ be the parameter vector. Then, the log-likelihood function for Θ, say $\ell=\ell(\Theta)$, is given by

$$\begin{aligned}\ell = \; & n\log\alpha + n\log\beta + n\log a + n\log b + (b-1)\sum_{i=0}^{n}\log x_i + (a\alpha-1)\sum_{i=0}^{n}\log(1-\tau_i) \\ & -(\alpha+1)\sum_{i=0}^{n}\log\left[1-(1-\tau_i)^a\right] - (1+\beta)\sum_{i=0}^{n}\log\left\{1+\left[\frac{(1-\tau_i)^a}{1-(1-\tau_i)^a}\right]^{\alpha}\right\}, \quad (11)\end{aligned}$$

where $\tau_i = e^{-x_i^b}$. Equation (11) can be maximized either directly by using ready software packages such as the `SAS` (`PROC NLMIXED`), `R` (`optim` function) or `Ox` program (sub-routine `MaxBFGS`) or by solving the nonlinear likelihood equations obtained by differentiating (11). The score vector components, say $\mathbf{L}(\Theta)=\frac{\partial\ell}{\partial\Theta}=(\mathbf{L}_\alpha, \mathbf{L}_\beta, \mathbf{L}_a, \mathbf{L}_b)^T$, are available from the corresponding author. Via setting the nonlinear system $\mathbf{L}_\alpha, \mathbf{L}_\beta, \mathbf{L}_a, \mathbf{L}_b = 0$ and solving them simultaneously yields the MLE $\widehat{\Theta}=(\widehat{\alpha}, \widehat{\beta}, \widehat{a}, \widehat{b})^T$ of $\Theta=(\alpha, \beta, a, b)^T$. If those equations cannot be solved analytically, the Newton-Raphson type algorithms can be used to solve them numerically. For the CIs estimation of the unknown model parameters, we should get the observed information matrix $J(\Theta)$ which is the output after using the above maximization procedures. When $n\to\infty$ (under standard regularity conditions), the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_4(0, J(\widehat{\Theta})^{-1})$ distribution to construct approximate CIs for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. The re-sampling bootstrap method may be adopted for correcting the biases of the MLEs of the model parameters. The CIs estimates may also be obtained using the bootstrap percentile method.

4. Real data applications

In this section, we will provide two real data applications to illustrate importance of the BXIIEW model. The MLEs of the parameters for all models are calculated and two goodness-of-fit statistics are used to compare the new model with other models. We shall compare the fits of the BXIIEW distribution with other competitive models such as the Weibull (W) (see [14]), the exponetiated Weibull (EW) (see [8] and [9]), the transmuted Weibull (TW) (see [2]), the beta Weibull (BW) (see [6]), the Burr X Weibull (BXW) (see [12]), the Kumaraswamy Weibull (KwW) (see [4], OLiW (see [13] and [1]), WGW (see [15]) and the McDonald Weibull (McW) (see [3]) distributions given by:

- WG-W :

$$f(x;\gamma,\theta,b)=\gamma\theta b x^{b-1}\left[1-\mathrm{e}^{-\theta x^{b}}\right]^{\gamma-1}\mathrm{e}^{-(\theta\gamma+2)x^{b}}e^{-\left[\frac{1-\mathrm{e}^{-\theta x^{b}}}{\mathrm{e}^{-\theta x^{b}}}\right]^{\gamma}},$$

- W :

$$f(x)=bx^{b-1}e^{-x^{b}},$$

- EW :

$$f(x)=abx^{b-1}e^{-x^{b}}\left(1-e^{-x^{b}}\right)^{a-1},$$

- OLiW

$$f(x)=\alpha^{2}\left(1+\alpha\right)^{-1}bx^{b-1}\,e^{2x^{b}}\;e^{-\alpha\frac{1-e^{-x^{b}}}{e^{-x^{b}}}}.$$

- BXW

$$f(x)=\;2\theta bx^{b-1}\left(1-\mathrm{e}^{-x^{b}}\right)\left(1-\mathrm{e}^{-\left[\mathrm{e}^{x^{b}}-1\right]^{2}}\right)^{\theta-1}\mathrm{e}^{2x^{\beta}-\left(\mathrm{e}^{x^{b}}-1\right)^{2}},$$

- BW :

$$f(x)=bx^{b-1}e^{-\gamma x^{b}}\left(1-e^{-x^{b}}\right)^{\alpha-1}/B(\alpha,\gamma),$$

- KwW :

$$f(x)=\alpha\gamma bx^{b-1}\left(1-e^{-x^{b}}\right)^{\alpha-1}\left[1-\left(1-e^{-x^{b}}\right)^{\alpha}\right]^{\gamma-1}e^{-x^{b}},$$

- TW :

$$f(x)=bx^{b-1}\left[1+\lambda-2\lambda\left(1-e^{-x^{b}}\right)\right]e^{-\alpha x^{b}},$$

- McW :

$$f(x)=\gamma bx^{b-1}e^{-ax^{b}}\left(1-e^{-x^{b}}\right)^{\alpha\gamma-1}\left[1-\left(1-e^{-x^{b}}\right)^{\gamma}\right]^{a-1}/B(\alpha,a).$$

The1^{st} data set consists of the failure times for a particular windshield model including 88 observations that are classified as failed times of windshields. These data were previously studied by [10]. Figure 3 gives the TTT plots for the 1stdata set, from Figure 3 we onclude that the empirical

HRFs of the data can be increasing.

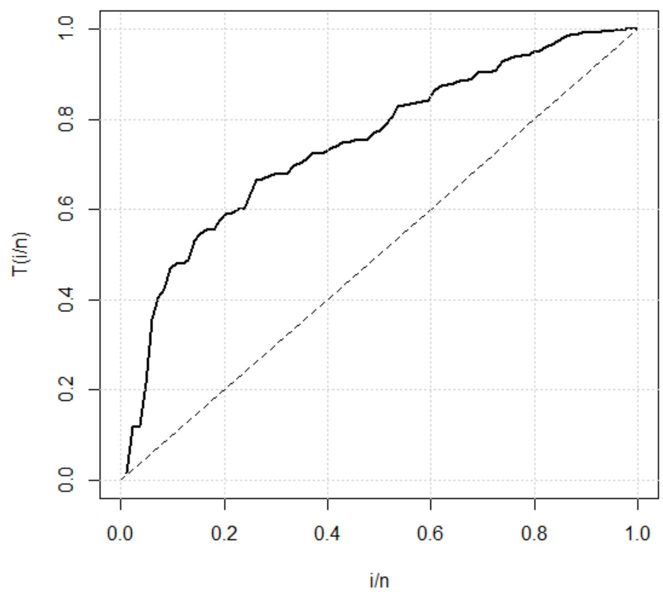

Figure 3: TTT plot for dta set **I**.

The 2^{nd} real data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 ([7]). The data was examined by [11]. Figure 4 gives the TTT plots for the 1stdata set, from Figure 4 we onclude that the empirical HRFs of the data can be unimodal.

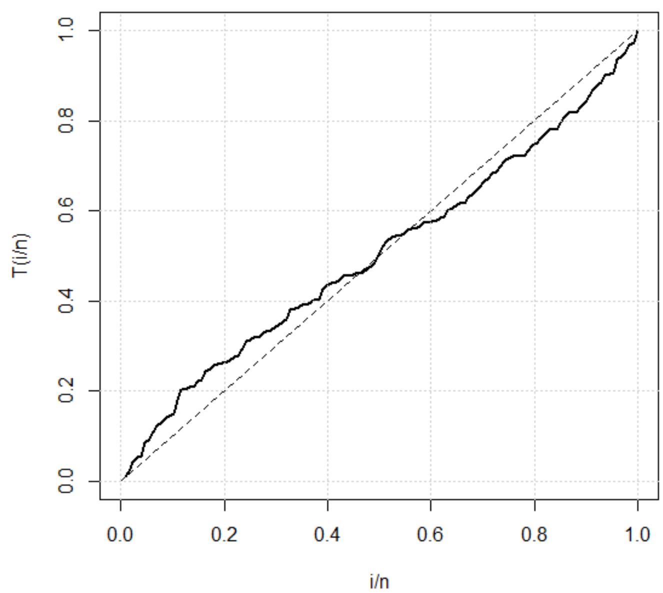

Figure 4: TTT plot for dta set **II**.

In order to compare the fitted models, we will consider some goodness-of-fit measures including the Akaike information criterion (**AIC**) and Bayesian information criterion (**BIC**) as

$$\mathbf{AIC} = 2p - 2\hat{\ell} \text{ and } \mathbf{BIC} = p\,\log\left(n\right) - 2\hat{\ell},$$

where p is the number of parameters, n is the sample size and $\hat{\ell}$ is the log-likelihood function evaluated at the MLEs. The smaller are values of these statistics, the better are the fits.Tables 1 and 2 list the MLEs of the model parameters and the numerical values of the model selection statistics **AIC** and**BIC** and K-S. We note from the figures in Table 1 that the BXIIEW model has the lowest values of the **AIC** and **BIC** (for the first data set) as compared to other models. The fitted PDF, HRF and P-P plot of the first data of the BXIIEW model is displayed in Figure 5. Similarly, it is also evident from Table 2 that the BXIIEW gives the lowest values the **AIC**, **BIC** (for the second data set) as compared to other models. The fitted PDF, HRF and P-P plot of the second data of the BXIIEW distribution is displayed in Figure 6.

Table 1: The MLEs and the goodness-of-fit statistics for the first data set.

Distribution	Parameter Estimates	**AIC**	**BIC**
W(b)	2.562	331.9	334.4
BXW(θ, b)	36.3, 0.123	314.5	319.4
EW(a, b)	3.595,1.316	286.7	291.6
BW(b, α, γ)	1464.1, 3.52, 2014.8	282.7	290.1
OLiW(α, b)	4.39, 0.63	280.5	285.4
KwW(b, α, γ)	80.66, 2.41, 3351.1	268.9	276.2
TW(b, λ)	1.749,−0.996	297.6	302.5
McW(γ, a, α, b)	27.80, 8.68, 0.256, 3.73	269.2	279.0
BXIIEW(α, β, a, b)	0.76, 0.11, 0.67, 2.33	**263.6**	**273.3**

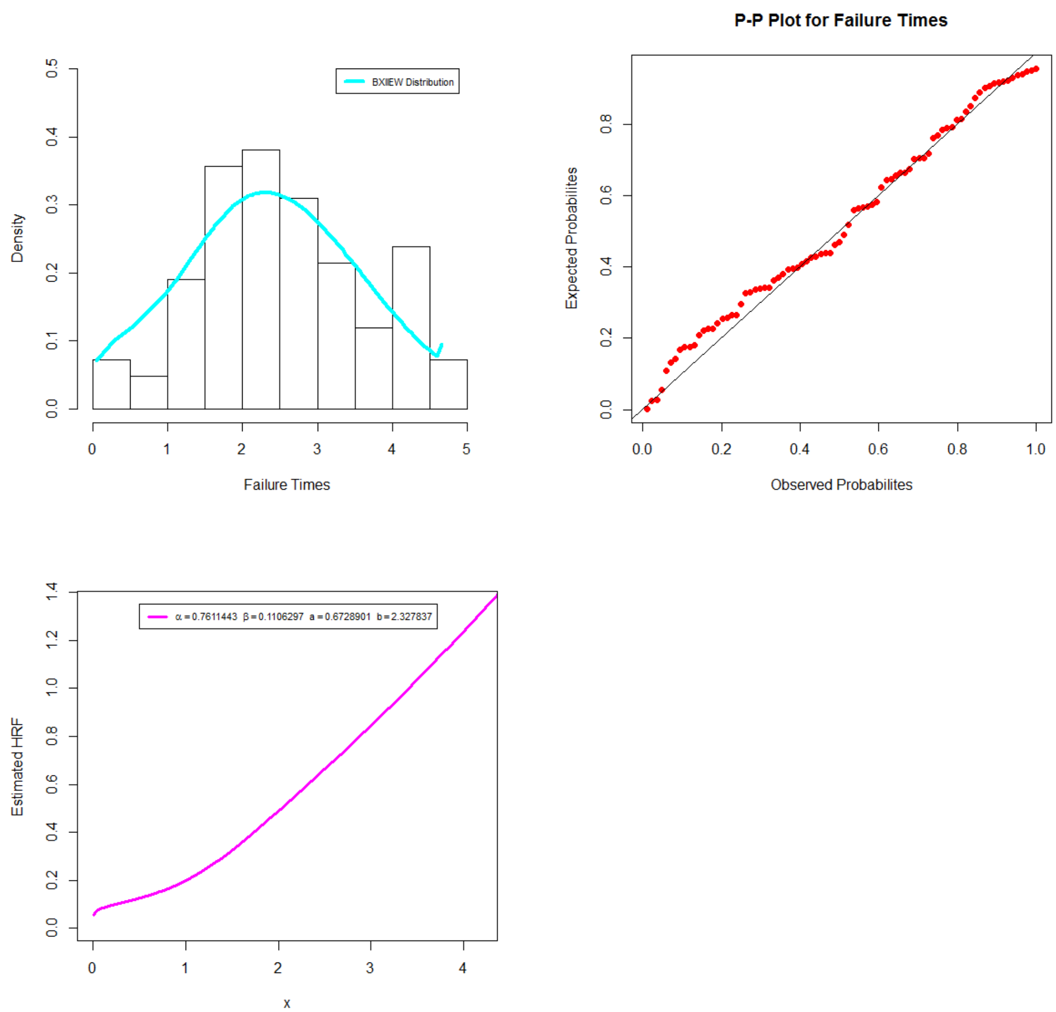

Figure 5: Estimated PDF, P-P plot and estimated HRF for the first data set.

Table 2: The MLEs and the goodness-of-fit statistics for the second data set.

Distribution	Parameter Estimates	**AIC**	**BIC**
W(b)	46.35	1172.2	1175.1
EW(a, b)	36.03, 1.515	1166.1	1173.7
BW(b, α, γ)	12635.4, 1.492, 406.1	1165.6	1173.9
McW(γ, a, α, b)	9.05, 2.28, 0.508, 169.4	1166.0	1177.2
WGW(θ, γ, b)	0.126, 0.957,10.06	1165.5	1172.8
BXIIEW(α, β, a, b)	0.23, 0.73, 0.08, 0.81	**1025.2**	**1036.6**

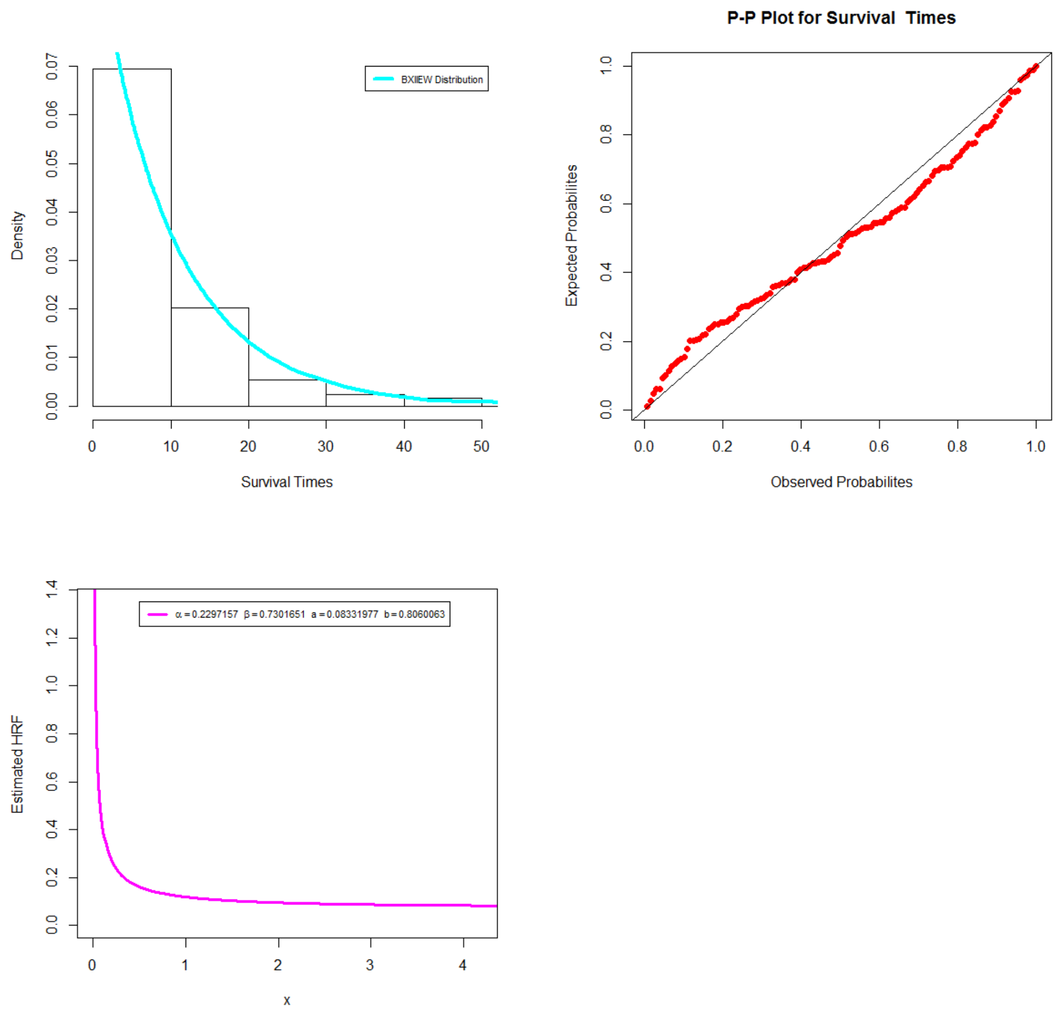

Figure 6: Estimated PDF, P-P plot and estimated HRF for the second data set.

5. Conclusions

In this work, a new extension of the exponentiated Weibull model with variable shapes of hazard rates is introduced and studied. Some of its mathematical properties are derived. Two applications are allocated to show the importance and flexibility of the new model. . We prove empirically the importance and flexibility of the new model in modeling two types of lifetime data, the new model has the lowest values of the **AIC** and **BIC** (for the 1^{st} data set). Similarly, the new model gives the lowest values the **AIC**, **BIC** (for the 2^{nd} data set) as compared to other models. The new model is much better than other competitive models.

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